Math 10B - Calculus of Several Variables II - Winter 2011
March 9, 2011
Practice Final

Name: Solutions

There is no need to use calculators on this exam. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, and an exam. All answers should be given as exact, closed form numbers as opposed to decimal approximations (i.e. $\pi$ as opposed to 3.14159265358979...). Cheating is strictly forbidden. You may leave when you are done. Good luck!

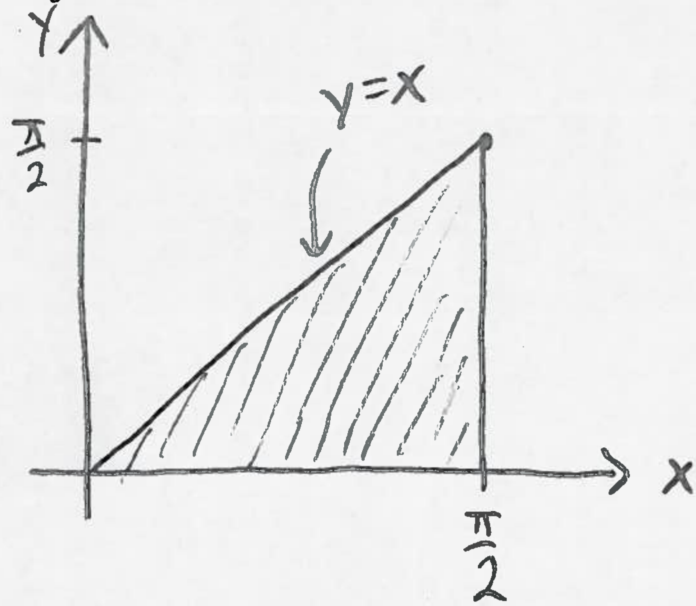
| Problem | Score |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /20 |
| 4 | /20 |
| 5 | /20 |
| 6 | /20 |
| 7 | /20 |
| 8 | /10 |
| 9 | /20 |
| 10 | /20 |
| Score | /170 |

**Problem 1** (10 points). *Compute the following integral:*

$$\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \sin x^2 \, dx dy.$$

*Draw the region of integration.*

Sketch:




$$\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \sin x^2 \, dx \, dy = \int_0^{\frac{\pi}{2}} \int_0^{x} \sin x^2 \, dy \, dx$$

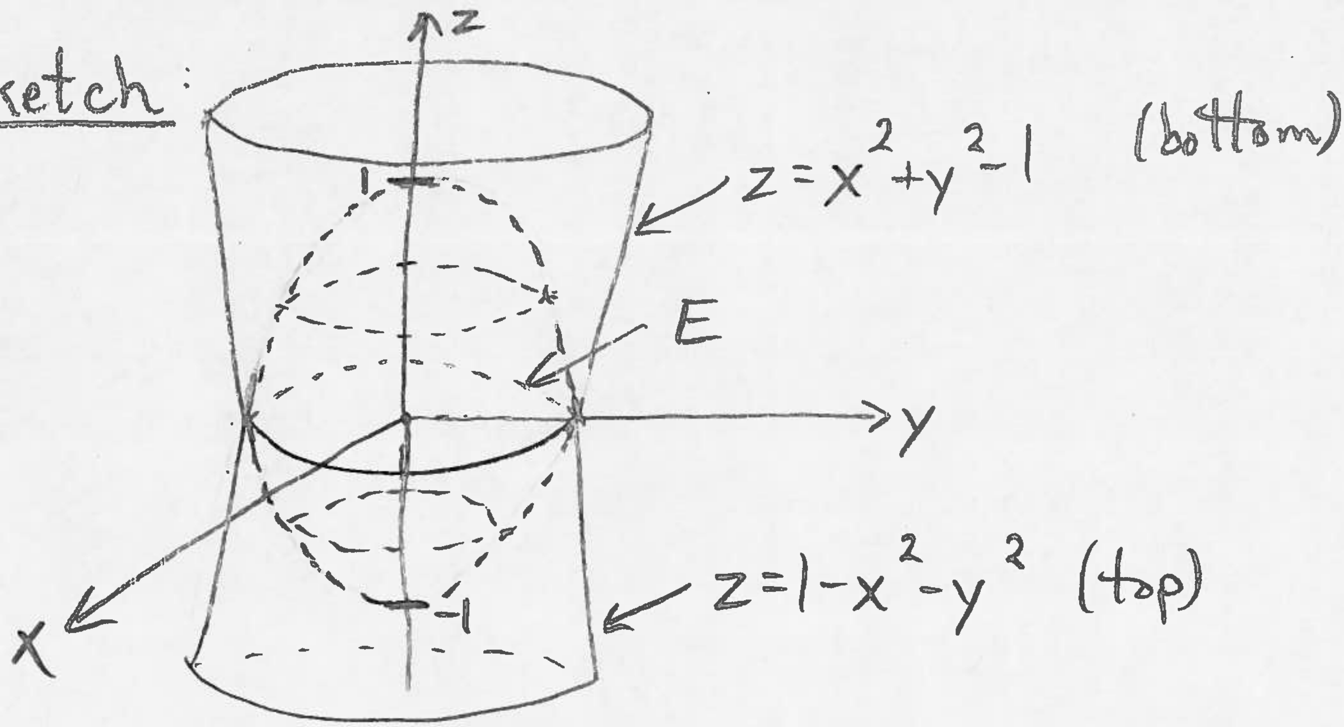
$$= \int_0^{\frac{\pi}{2}} y \sin x^2 \Big|_0^x dx = \int_0^{\frac{\pi}{2}} x \sin x^2 \, dx \qquad \begin{array}{l} u = x^2 \\ du = 2x dx \end{array}$$

$$= \int_0^{\frac{\pi^2}{4}} \frac{1}{2} \sin u \, du = \frac{-1}{2} \cos u \Big|_0^{\frac{\pi^2}{4}} = \frac{-\cos \frac{\pi^2}{4}}{2} + \frac{\cos 0}{2}$$

$$= \boxed{\frac{1}{2}\left(1 - \cos \frac{\pi^2}{4}\right)}$$

**Problem 2** (10 points). *Find the volume of the region bounded by* $z = x^2+y^2-1$ *and* $z = 1-x^2-y^2$.

Sketch:




In cylindrical, this region is

$0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ r^2-1 \le z \le 1-r^2$

$$\text{Vol} = \int_0^{2\pi}\int_0^1\int_{r^2-1}^{1-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\left[(1-r^2)-(r^2-1)\right] dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[2r-2r^3\right] dr\,d\theta = \int_0^{2\pi} \left(r^2-\tfrac{1}{2}r^4\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\,d\theta = \boxed{\pi}$$

**Problem 3** (20 points).

(a) *(10 points) Compute the Jacobian* $\frac{\partial(x,y)}{\partial(r,\theta)}$ *for changing Cartesian coordinates to polar coordinates.*

(b) *(10 points) Let* $D$ *be the region bounded by* $x^2+y^2=5$ *where* $x \geq 0$. *Compute the integral*

$$\iint_D e^{x^2+y^2}\,dA.$$

(a) $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

(b) $$\iint_D e^{x^2+y^2}\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\sqrt{5}} e^{r^2}\, r\,dr\,d\theta \qquad u = r^2 \quad du = 2r\,dr$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{5} e^{u}\,du\,d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^5-1)\,d\theta$$

$$= \boxed{\frac{\pi(e^5-1)}{2}}$$

**Problem 4** (20 points).

(a) *(10 points) Parametrize the circle of radius $r$.*

(b) *(10 points) Use this parametrization to show that the circumference of the circle of radius $r$ is $2\pi r$. (Hint: Use arclength.)*

(a) $\boxed{\vec{r}(\theta) = \langle r\cos\theta, r\sin\theta\rangle, \quad 0 \le \theta \le 2\pi}$

(b) $$\int_{C_r} ds = \int_0^{2\pi} |\vec{r}\,'(\theta)|\,d\theta = \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\,d\theta$$

$$= \int_0^{2\pi} r\,d\theta = 2\pi r.$$

**Problem 5** (20 points). *Let $C$ be the boundary of the region bounded by $y = x^2$ and $x = y^2$, oriented counterclockwise.*

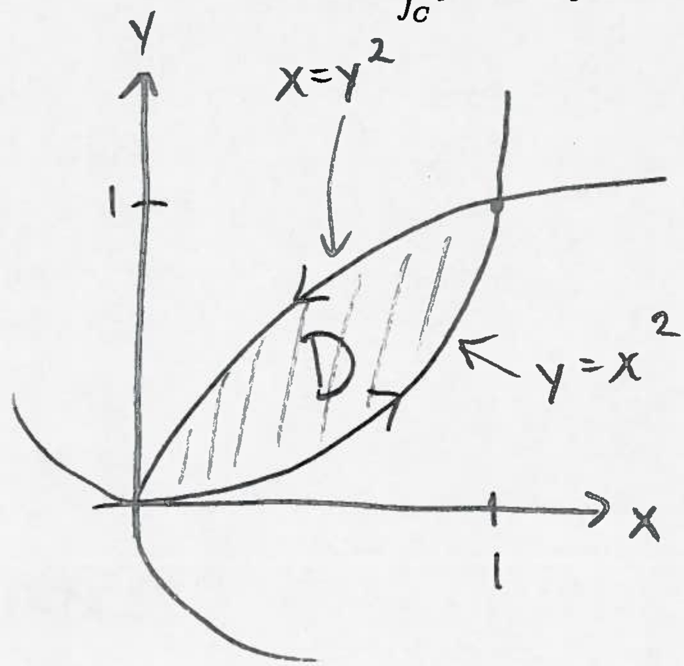
(a) *(10 points) Compute the integral*

$$\int_C \arctan x^3 \, dx + \ln(y^2+1) \, dy.$$

(b) *(10 points) Compute the integral*

$$\int_C y \, dx - x \, dy.$$

Sketch:




ⓐ Since $C$ has positive orientation, Green's theorem gives $\int_C \arctan(x^3)dx + \ln(y^2+1)\,dy = \iint_D (0-0)\,dA = \boxed{0}$

ⓑ Again, by Green's theorem:

$$\int_C y\,dx - x\,dy = \iint_D (-1-1)\,dA = \int_0^1 \int_{y^2}^{\sqrt{y}} -2\,dx\,dy$$

$$= \int_0^1 (-2\sqrt{y} + 2y^2)\,dy = \frac{-4}{3}y^{3/2} + \frac{2}{3}y^3 \Big|_0^1 = \boxed{\frac{-2}{3}}$$

**Problem 6** (20 points). *Determine whether the following vector fields are conservative. Find a scalar potential function for the ones that are conservative.*

(a) *(10 points)*

$$\vec{\mathbf{F}}(x,y) = (\underset{P}{2x\sin y}, \underset{Q}{x^2\cos y}).$$

(b) *(10 points)*

$$\vec{\mathbf{G}}(x,y,z) = (\underset{P}{y+z}, \underset{Q}{2z}, \underset{R}{x+y}).$$

ⓐ If $\vec{F} = \nabla f = \langle f_x, f_y \rangle$

$f = \int P\,dx = \int 2x\sin y\,dx = x^2\sin y + c(y)$

$f_y = x^2\cos y + c'(y) = Q = x^2\cos y \Rightarrow c'(y) = 0 \Rightarrow c(y) = \text{const}$

So, $\boxed{f = x^2\sin y}$ is a potential, and hence $\vec{F}$ is conservative.

ⓑ If $\vec{G} = \nabla g = \langle g_x, g_y, g_z \rangle$

$g = \int P\,dx = \int (y+z)\,dx = xy + xz + c(y,z)$

$g_y = x + c_y(y,z) = Q = 2z$

But this cannot happen since $c_y$ does not depend on $x$.

So

$$\text{curl}\,\vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & 2z & x+y \end{vmatrix} = \langle 1-2, 1-1, 0-1 \rangle = \langle -1, 0, -1 \rangle \neq \vec{0}$$

So, $\boxed{\vec{G} \text{ is not conservative}}$

**Problem 7** (20 points). *Let $f$ be a $C^1$ function on some region $D \subset \mathbb{R}^2$, and consider the surface given by $z = f(x, y)$. Show that the surface area of this surface is given by*

$$S.A. = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

*Hint: Recall that surface area is given by*

$$S.A. = \iint_{\mathbf{X}} dS$$

*where $\mathbf{X}$ is a parametrization of the surface.*

A parametrization of the surface $S$ is

$$\vec{X}(x,y) = \langle x, y, f(x,y) \rangle$$

$$\vec{X}_x = \langle 1, 0, f_x \rangle, \quad \vec{X}_y = \langle 0, 1, f_y \rangle$$

$$\vec{X}_x \times \vec{X}_y = \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{X}_x \times \vec{X}_y| = \sqrt{f_x^2 + f_y^2 + 1}$$

$$S.A. = \iint_S dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

**Problem 8** (10 points). *Let $S$ denote the closed cylinder with bottom given by $z = 0$, top given by $z = 7$, and lateral surface given by $x^2 + y^2 = 49$. Orient $S$ with outward normals. Compute the following integral:*

$$\iint_S (-y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{S}.$$

Recall the divergence theorem:
Since $S$ is oriented with outward normals

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\operatorname{div} \vec{F})\, dV$$

where $S = \partial E$.

$$\operatorname{div}(-y\hat{\imath} + x\hat{\jmath}) = 0+0+0 = 0$$

So

$$\iint_S (-y\hat{\imath} + x\hat{\jmath}) \cdot d\vec{S} = \iiint_E 0 \, dV = \boxed{0}$$

**Problem 9** (20 points). *Let $S$ be the sphere given by $x^2+y^2+z^2=1$ with outward pointing normals.*

(a) *(10 points) Let* $\mathbf{F}(x,y,z)=(2xyz+5z, e^x\cos yz, x^2y)$. *Compute*

$$\iint_S \mathit{curl}\,\mathbf{F}\cdot d\mathbf{S}.$$

(b) *(10 points) Let* $\mathbf{G}(x,y,z)=(x,y,z)$. *Compute*

$$\iint_S \mathbf{G}\cdot d\mathbf{S}.$$

*Hint: The volume of a sphere of radius $r$ is given by $V=\frac{4}{3}\pi r^3$.*

ⓐ Recall that if $C=\partial S$ has orientation consistent with $S$, then Stokes' theorem gives

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}=\int_C \vec{F}\cdot d\vec{r}$$

Here $S$ has no boundary since $S$ is closed.

So $$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}=\int_{\emptyset}\vec{F}\cdot d\vec{r}=\boxed{0}$$

($\emptyset$ = empty set = "nothing")

ⓑ $S$ is positively oriented, so we can use the divergence theorem ($S=\partial E$)

$$\iint_S \vec{G}\cdot d\vec{S}=\iiint_E (\operatorname{div}\vec{G})\,dV=\iiint_E (1+1+1)\,dV$$

$$=3\cdot \operatorname{vol}(E)=3\cdot\left(\tfrac{4}{3}\pi(1)^3\right)=\boxed{4\pi}$$

**Problem 10** (20 points). *Verify that Stokes' theorem implies Green's theorem. Hint: Use the vector field* $\mathbf{F}(x,y,z) = (M(x,y), N(x,y), 0)$.

Let C be a piecewise smooth, simple, positively oriented, plane curve and let S be the region it bounds in the xy-plane. Consider these all to be in $\mathbb{R}^3$ by letting the z-component be zero. Give S the upward orientation, which in this is exactly $\vec{n} = \langle 0,0,1 \rangle$ since S is entirely in the xy-plane. Let $\vec{F}(x,y,z) = \langle M(x,y), N(x,y), 0 \rangle$ be $C'$ (i.e., $M$ & $N$ are $C'$). Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \langle M, N, 0 \rangle \cdot \langle dx, dy, dz \rangle = \int_C M\,dx + N\,dy$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \langle 0, 0, N_x - M_y \rangle$$

$$\iint_S (\text{curl } \vec{F}) \cdot d\vec{S} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, dS = \iint_S (N_x - M_y)\, dS$$

and since the surface is flat, $dS = dA$ and:

$$\int_C M\,dx + N\,dy = \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_S (N_x - M_y)\, dA$$